

Conservation and persistence of spin currents and their relation to the Lieb-Schulz-Mattis twist operators

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Systems with spin-orbit coupling do not conserve “bare” spin current \mathbf{j} . A recent proposal for a conserved spin current \mathbf{J} [J. Shi *et al.*, Phys. Rev. Lett. **96**, 076604 (2006)] does not flow persistently in equilibrium. We suggest another conserved spin current $\bar{\mathbf{J}}$ that may flow persistently in equilibrium. We give two arguments for the instability of persistent current of the form \mathbf{J} : one based on the equations of motions and another based on a variational construction using Lieb-Schulz-Mattis twist operators. In the absence of spin-orbit coupling, the three forms of spin current coincide.

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I. INTRODUCTION

Spintronics aims to manipulate the electron’s spin instead of its charge.^{1–4} A requisite for spintronic devices is the generation of a spin current.^{5,6} In recent years, exciting new proposals and predictions have been made regarding the possibility of dissipationless spin currents primarily through the use of the spin Hall effect (SHE).^{7,8} The SHE may generate sizable Hall spin currents and open the door for the realization of many of these notions. Observation of the SHE have been reported by employing the Kerr effect in *n*-doped GaAs (Ref. 9) and a *p*-*n* junction light-emitting diode system.¹⁰ Much fundamental work on spin currents has been carried out, e.g., Refs. 11–16.

A fundamental problem in spintronics is the nonconservation of the bare spin currents [the sum of the product of each electron spin multiplied by its velocity, see Eq. (3)]. This nonconservation is triggered by spin-orbit effects. Charge conservation and the continuity equations that it gives rise to are of crucial importance in electronic circuits. Is there an analog for spins? The quest for conserved spin currents has been the subject of much attention.^{17–19} An early step was made in Ref. 17 that proposed a conserved spin current by augmenting the bare spin currents by an additional contribution so as to include the effects of torques triggered by spin-orbit effects. The work in Ref. 17 assumed that the average torque vanishes in the bulk and that the “torque dipole density” on the system’s surface can also be neglected. The torque dipole density was a field whose divergence led to the actual torque acting on spins at a particular location. This proposal was exciting. The spin-accumulation observations in Refs. 9 and 10 indeed cannot measure the bare spin current.

We suggest that the notions of conservation and of persistence are independent of spin currents. For example, the bare spin current in Eq. (3) in general is not conserved and may persist in equilibrium [see Eq. (10)]. Rashba noticed the possibility of persistent equilibrium flow of the bare spin current for a particular class of spin-orbit coupled models.²⁰ Here we present a model-independent argument for the result. In contrast to the bare spin current, the conserved spin current \mathbf{J} given by Eq. (2) does not flow in energy eigenstates so any persistent flow implies the system is not in equilibrium. Fur-

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thermore, we have found another conserved spin current $\bar{\mathbf{J}}$ and associated spin density $\bar{\rho}$ that may flow persistently. When spin is a good quantum number, these distinctions collapse: the bare spin current is conserved and none of the three spin currents flow persistently. However, in a generic, spin-orbit coupled system, the definitions of spin current differ with respect to their conservation and persistence.

In Sec. II, we recall the standard definition of spin density and bare spin current. We then review the conserved spin current proposed in Ref. 17 and contrast it with an alternative, conserved spin current [see Eq. (5)]. Sec. III further distinguishes this novel, conserved current by computing its expectation in a general spin-orbit coupled system. We show that, in contrast to the previous proposal, it may flow persistently, just like the bare spin current. In Sec. IV we give a physical argument to explain this distinction: any state that carries the conserved spin current proposed in Ref. 17 may dissipate energy by relaxing to a lower-energy state, which we explicitly construct. The argument extends the one used by Bohm to establish that persistent charge current flows only in metastable states, not the ground state.²¹ Further, the construction generalizes one used by Lieb *et al.*²² to study quasi-one-dimensional, spin-1/2 systems. We conclude with a summary and discussion of our results.

II. DEFINITIONS OF SPIN CURRENT AND SPIN DENSITY

Let us consider a general, interacting system with Hamiltonian $H[\{\mathbf{r}_i\}, \{\mathbf{p}_i\}, \{\sigma_{\alpha}\}]$. Here, $\{\mathbf{r}_i\}$ and $\{\mathbf{p}_i\}$ denote the phase-space variables and $\{\sigma_{\alpha}\}$ reflects the spin states of all nonscalar particles in the system. The spin density along a given quantization axis (taken to be the *z* axis) has the form

$$\rho = \sum_i \sigma_i^z n_i, \quad (1)$$

where n_i is the particle density and σ_i^z is the spin operator along the *z* axis, given by a Pauli matrix for spin one-half particles. In Ref. 17, Shi *et al.* suggested the following form for the spin current:

$$\mathbf{J} = \frac{d}{dt} \left(\sum_i \sigma_i^z \mathbf{x}_i \right) = \mathbf{j} + \sum_i \dot{\tau}_i^z \mathbf{x}_i, \quad (2)$$

where the spin torque is given by $\dot{\tau}_i^z = d\sigma_i^z/dt = 1/(i\hbar)[\sigma_i^z, H]$. In the absence of spin-orbit coupling, the spin torque van-

ishes and the spin current \mathbf{J} coincides with the bare spin current

$$\mathbf{j} = \sum_i \sigma_i^z \mathbf{v}_i, \quad (3)$$

where the velocity is given by $\mathbf{v}_i = d\mathbf{x}_i/dt = 1/(i\hbar)[\mathbf{x}_i, H]$.

In contrast to the bare spin current Eq. (3), the current in Eq. (2) satisfies the continuity equation, even in systems with spin-orbit coupling¹⁷

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \quad (4)$$

Physically this means that in a system with boundary, the flow of spin current Eq. (2) across the boundary into the system leads to spin accumulation within the system. We note that conservation of \mathbf{J} only occurs when the average torque vanishes: $\sum_i \langle \tau_i^z \rangle = 0$.¹⁷

Interestingly, it is possible to construct another spin current $\bar{\mathbf{J}}$ and spin density $\bar{\rho}$ that satisfy the continuity equation. Consider the following spin current:

$$\bar{\mathbf{J}} = \sum_i \sigma_i^z(t=0) \mathbf{v}_i = \mathbf{j} - \sum_i \tau_i^z \mathbf{x}_i + \dots, \quad (5)$$

where, the omitted terms are higher order in the spin-orbit coupling. We define a corresponding spin density $\bar{\rho} = \sum_i \sigma_i^z(t=0) n_i$ by evaluating at the same reference time. Remarkably, this form of the spin density and spin current also satisfies the continuity equation

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot \bar{\mathbf{J}} = 0. \quad (6)$$

To see this, notice that the label $\sigma_i^z(t=0)$ does not evolve but merely labels each particle. From the continuity of charge, the continuity of these labels follows. We remark that the current $\bar{\mathbf{J}}$ obeys the continuity Eq. (6) even when the average torque does not vanish, in contrast to the current \mathbf{J} . In the Appendix, we perform the expansion in spin-orbit coupling explicitly.

III. CONSTRAINTS ON SPIN CURRENT FROM EQUATIONS OF MOTION

As noted in Ref. 17, the form of the spin current in Eq. (2) does not flow in energy eigenstates. We give a simplified derivation, then extend it to finite temperature, and use the result to deduce constraints on equilibrium flow of the other two forms of spin current Eqs. (3) and (5).

The spin displacement operator

$$\mathbf{L} = \sum_i \mathbf{x}_i \sigma_i^z \quad (7)$$

allows us to evaluate the spin current in Eq. (2),

$$\langle n | \mathbf{J} | n \rangle = \langle n | \frac{d\mathbf{L}}{dt} | n \rangle = -i \langle n | [\mathbf{L}, H] | n \rangle = 0. \quad (8)$$

Here, $|n\rangle$ denotes the n th eigenstate of H with energy E_n . The second equality in Eq. (8) follows from the equations of motion while the third follows as $\langle n |$ and $|n\rangle$ refer to the same eigenstate of H with specific eigenvalue (E_n). Here and henceforth, we set $\hbar = 1$.

As an immediate corollary, we notice that the spin current in Eq. (2) vanishes at finite temperatures as well

$$\langle \mathbf{J} \rangle = \frac{\sum_n \langle n | \mathbf{J} | n \rangle e^{-\beta E_n}}{\sum_n e^{-\beta E_n}} = 0, \quad (9)$$

where, the inverse temperature is $\beta = 1/k_B T$.

If we now invoke Eqs. (8) and (9), we have the following constraint on the bare spin current Eq. (3) in equilibrium:

$$\langle \mathbf{j} \rangle = - \sum_i \langle \tau_i^z \mathbf{x}_i \rangle. \quad (10)$$

This is so due to Eq. (9) and the relation between Eqs. (2) and (3). Similarly, the conserved current $\bar{\mathbf{J}}$ may have a non-vanishing expectation value even at equilibrium

$$\langle \bar{\mathbf{J}} \rangle = -2 \sum_i \langle \tau_i^z \mathbf{x}_i \rangle + \dots, \quad (11)$$

where the omitted terms are higher order in the spin-orbit coupling. When spin is a good quantum number, all three forms of the spin current vanish in equilibrium. Only in the presence of spin-orbit coupling do the equilibrium currents differ.

IV. VARIATIONAL PROOF USING THE LIEB-SCHULZ-MATTIS TWIST OPERATOR

We now regress and derive a version of Eq. (8) for the ground-state sector by a generalization of the variational arguments in Ref. 21. This proof makes contact with topological Lieb-Schulz-Mattis²² type twist operators.

We consider the ground state $|0\rangle$ of a general system with spin-orbit coupling and prove by a variational argument that within it the spin current vanishes

$$\langle 0 | \mathbf{J} | 0 \rangle = 0, \quad (12)$$

where, we take spin current \mathbf{J} as in Eq. (2). By definition, the ground-state energy

$$E_0 \equiv \langle 0 | H | 0 \rangle \quad (13)$$

is the lowest attainable energy and lies at the bottom of the spectrum. Let us assume that we have a state $|\psi\rangle$ with a finite (i.e., nonzero) spin current. We will then prove that $|\psi\rangle$ cannot be the ground state, by constructing another state $|\psi(\delta\mathbf{P})\rangle$ with a lower energy. That is, if $\langle \psi | \mathbf{J} | \psi \rangle \neq 0$ then, the energy $E(\delta\mathbf{P})$ of the “twisted” state $|\psi(\delta\mathbf{P})\rangle$ can be made to be lower than the energy $E(0)$ of $|\psi\rangle$, by choosing the variational parameter $\delta\mathbf{P}$ appropriately

$$E(\delta\mathbf{P}) < E(0). \quad (14)$$

In this way we establish that the ground state $|0\rangle$ cannot carry a spin current of the form in Eq. (2) since, by definition, it has the lowest energy.

We construct the variational state $|\psi(\delta\mathbf{P})\rangle$ by applying a spin-dependent boost to the state $|\psi\rangle$

$$|\psi\rangle = \exp(-i\delta\mathbf{P} \cdot \mathbf{L}) |\psi\rangle = \exp\left(-i\delta\mathbf{P} \cdot \sum_i \mathbf{x}_i \sigma_i^z\right), \quad (15)$$

where we have used the spin-displacement operator \mathbf{L} from Eq. (7). Compared to $|\psi\rangle$, the momentum of each particle in $|\psi(\delta\mathbf{P})\rangle$ with spin up $\sigma_i^z = +S$ changes by $+S\delta\mathbf{P}$ while the momentum of particles with spin down $\sigma_i^z = -1$ changes by $-S\delta\mathbf{P}$. The momentum boost $\delta\mathbf{P}$ will be our variational parameter. Lieb-Schulz-Mattis perform a similar construction but restrict the system geometry to be quasi-one-dimensional

and the spin S to be half integer.²² We require no such restrictions; the conclusion Eq. (12) holds in one, two, and three dimensions, for systems with half-integer spin particles as well as those containing integer spin particles.

The energy of $|\psi(\delta\mathbf{P})\rangle$ for small δP may be expanded

$$\begin{aligned} \langle \psi(\delta\mathbf{P})|H|\psi(\delta\mathbf{P})\rangle &= \langle \psi|H|\psi\rangle - i\delta\mathbf{P} \cdot \langle \psi| \left[H, \sum_i \sigma_i^z \mathbf{x}_i \right] |\psi\rangle \\ &+ \mathcal{O}[(\delta P)^2]. \end{aligned} \quad (16)$$

Now invoking the equation of motion $\langle d\mathbf{L}/dt\rangle = i\langle [H, \mathbf{L}]\rangle$ and the definition of the spin current $\mathbf{J} = d\mathbf{L}/dt$ in Eq. (2) we have

$$E(\delta\mathbf{P}) = E(0) + \delta\mathbf{P} \cdot \langle \psi|\mathbf{J}|\psi\rangle + \mathcal{O}[(\delta P)^2]. \quad (17)$$

It follows that if the state $|\psi\rangle$ carries a spin current $\langle \psi|\mathbf{J}|\psi\rangle \neq 0$ then there always exists a sufficiently small δP such that $E(\delta\mathbf{P})$ can be made smaller than $E(0)$, by choosing the direction of the momentum boost for spin-up particles $\delta\mathbf{P}$ opposite to the current direction (\mathbf{J}). Since the ground state has the lowest energy, we must have Eq. (12), a special case of the more general Eq. (8).

V. CONCLUSIONS AND DISCUSSION

In spin-orbit coupled systems, the definition of spin current has inherent ambiguity.²³ By redefining the current, it is possible to change its properties. We consider three definitions of the spin current that generically differ with respect to the two properties that interest us most, continuity and persistence. The bare spin current \mathbf{j} defined in Eq. (3) may persist in equilibrium but generally does not lead to spin accumulation. The spin current \mathbf{J} defined in Ref. 17 obeys the conservation law Eq. (4) when the average torque vanishes. However, the expectation $\langle n|\mathbf{J}|n\rangle$ vanishes in all energy eigenstates $|n\rangle$ of a system with a time-independent Hamiltonian. A third definition Eq. (5) of the spin current $\bar{\mathbf{J}}$, obeys a continuity equation with a modified spin density $\bar{\rho}$ and may flow persistently in equilibrium.

Expanding in powers of the spin-orbit coupling, we find that $\bar{\mathbf{J}}$ differs at leading order from both the bare spin current \mathbf{j} and the conserved current \mathbf{J} in Ref. 17. This means that in principle, the expectation value $\langle n|\bar{\mathbf{J}}|n\rangle$ need not vanish and a conserved spin current $\bar{\mathbf{J}}$ may flow persistently even in equilibrium despite spin-orbit coupling. We conclude that in general spin-orbit coupled systems, no definite relation exists among the concepts of persistence and conservation of spin current.

We give a physical argument for the instability of persistent spin current of the form \mathbf{J} defined in Ref. 17. The argument makes contact with the Bloch's famous result for the instability of persistent charge currents. In passing, we note a fundamental relation, summarized in Eq. (15), between \mathbf{J} the spin current in Ref. 17 and the twist generator used by Lieb, Schulz, and Mattis. It would be interesting to formulate this relation in an $SU(2)$ invariant fashion.

Several experiments bear directly on these issues.^{24–26} Transport measurements in HgTe quantum wells provide evidence for gapless edge modes that are protected from nonmagnetic impurity scattering and that respond singularly to an external magnetic field.²⁵ This is consistent with a “heli-

cal” picture for the edge modes with a bare spin current \mathbf{j} accompanying the charge current. Spectroscopic measurements of the surface excitations in $\text{Bi}_{1-x}\text{Sb}_x$ find an odd number of gapless Dirac points on the surface, suggesting topological stability of the gapless modes to nonmagnetic impurities.²⁶ Further, in GaAs/GaAlAs quantum wells, time-resolved Kerr-rotation experiments see evidence for a long-lived excitation mode that transports both spin and charge.²⁴

These systems contain strong spin-orbit coupling, so the bare spin current \mathbf{j} accompanying the gapless charge modes need not give rise to a spin accumulation. A detectable change in spin density ρ , could arise in response to an applied bias due to the flow of the current \mathbf{J} , provided that the average torque density vanishes. For pure Rashba spin-orbit coupling, nonmagnetic impurity scattering eliminates spin accumulation even under an applied bias.¹⁵ The systems described above are stable to nonmagnetic impurities, opening up the possibility of detectable spin accumulation.

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APPENDIX: DERIVATION OF CONSERVED SPIN CURRENT $\bar{\mathbf{J}}$

We begin by considering the conservation of charge current. In the absence of spin-orbit coupling, this case also describes spin. Charge conservation and gauge invariance are tightly linked. To see this in the simplest context, we consider a system with approximate $d+1$ -dimensional Lorentz invariance such as graphene ($d=2$) or bismuth ($d=3$) and adopt the relativistic notation A_μ , where, $A_0 = \phi$ is the scalar potential and A_a for $a=1, 2, \dots, d$ are the components of the vector potential. For convenience, we adopt the units $\hbar=c=1$.

Let us now suppose that the low-energy effective action $S_{\text{eff}}[A_\mu]$ is invariant under a transformation of the gauge field

$$A_\mu \rightarrow A_\mu + \delta A_\mu = A_\mu - \partial_\mu \Lambda \quad (A1)$$

with $\Lambda(x_\mu)$ any smooth function. To see how this constrains charge flow, we formally expand the low-energy effective action in powers of the gauge field A_μ ,

$$S_{\text{eff}}[A_\mu] = \int d^d x dt J_c^\mu A_\mu + \dots, \quad (A2)$$

where, the charge current is given formally by the functional derivative $J_c^\mu = \frac{\delta S_{\text{eff}}[A_\mu]}{\delta A_\mu}$. By definition of the charge current, the gauge transformation Eq. (A1) leads to the following change in the effective action:

$$\delta S_{\text{eff}} = \int d^d x dt J_c^\mu \delta A_\mu = \int d^d x dt (\partial_\mu J_c^\mu) \Lambda + \dots, \quad (A3)$$

where, we have integrated by parts in the second line and dropped the contribution from the surface of the system. Gauge invariance requires $\delta S_{\text{eff}}=0$ for arbitrary, smooth Λ , which can only be satisfied if the charge current is conserved

$$\partial_\mu J_c^\mu = 0. \quad (\text{A4})$$

Conversely, charge conservation Eq. (A4) implies invariance of the effective action under the gauge transformation Eq. (A1). Thus, charge conservation and gauge invariance have an intimate relationship.

To make contact with our results in Sec. II, let us express this connection between gauge invariance and charge conservation in the language of operators. For a system with time-independent charges (e_i) the charge-density operator is given by

$$\rho_c = \sum_i e_i n_i \quad (\text{A5})$$

with n_i as the number-density operator. If we now assume gauge invariance and the usual Maxwell dynamics for the gauge field, then we find charge is conserved at the operator level,

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{J}_c = 0 \quad (\text{A6})$$

with the charge-current operator given as follows:

$$\mathbf{J}_c = \sum_i e_i \mathbf{v}_i. \quad (\text{A7})$$

Here \mathbf{v}_i is the velocity of the i th charge carrier in the system.

For example, consider a single particle with mass m and charge e in state $|\psi\rangle$. Symmetrizing Eq. (A7), we find the following familiar expression for the charge current (Here $a=1, 2, \dots, d$ run over the spatial components):

$$\langle \psi | J_c^a | \psi \rangle = e \int d^d x \left(\psi^* \left[\frac{1}{2mi} D_a \right] \psi - \psi \left[\frac{1}{2mi} D_a \right] \psi^* \right) \quad (\text{A8})$$

with the spatial components of the covariant derivative given by, $D_a = \frac{\partial}{\partial x^a} - ieA_a$. The velocity operator is given by $\mathbf{v} = \mathbf{D}/m$, and we see here that the operator expression (A7) describes the usual charge current in Eq. (A8).

Let us now apply this to spin systems. As the spin states change with time, we obviously cannot simply substitute $e_i \rightarrow \sigma_i^z(t)$ in order to get a conserved current. If we set, however, $e_i \rightarrow \sigma_i^z(t=0)$ then the steps above can be reproduced

word for word. Although, we will refer to spin currents in what follows, our results hold for any other ‘‘charge’’ field σ which is allowed to vary in time. This is so as $\sigma^z(0)$ the value of the z component of the spin at time $t=0$ is, by definition, time independent. We may relate $\sigma^z(0)$ to $\sigma^z(t)$ by a time-reversed evolution,

$$\sigma^z(0) = e^{-i\int_0^t H(t') dt'} \sigma^z(t) e^{i\int_0^t H(t') dt'}. \quad (\text{A9})$$

Here, $e^{i\int_0^t H(t') dt'}$ is a time-ordered exponential. The total number operator n_i (the sum over both up- and down-spin flavors) and its associated number current satisfies the continuity Eq. (A7), so the continuity equation for the spin current Eq. (5) given by Eq. (6) follows.

It is interesting to formally expand the spin density $\bar{\rho}$ in powers of the spin-orbit coupling,

$$\begin{aligned} \bar{\rho} = \sum_i \sigma_i^z(t=0) n_i &= \sum_i \sigma_i^z(t) n_i - t \sum_i \frac{d\sigma_i^z}{dt} n_i + \frac{t^2}{2} \sum_i \frac{d^2 \sigma_i^z}{dt^2} n_i \\ &\quad - \frac{t^3}{3!} \sum_i \frac{d^3 \sigma_i^z}{dt^3} n_i + \dots \end{aligned} \quad (\text{A10})$$

To zeroth order, $\bar{\rho}$ coincides with the bare spin density ρ . However, already at leading order the two densities differ.

Turning now to the spin current, we can use the expression in Eq. (A7) to write down a corresponding conserved spin current. Again it is interesting to formally expand the expression in powers of the spin-orbit coupling,

$$\begin{aligned} \bar{\mathbf{J}} = \sum_i \sigma_i^z(t=0) \mathbf{v}_i &= \sum_i e^{-iHt} \sigma_i^z(t) e^{iHt} \mathbf{v}_i \\ &= \sum_i \sigma_i^z \mathbf{v}_i - t \sum_i \frac{d\sigma_i^z}{dt} \mathbf{v}_i + \frac{t^2}{2} \sum_i \frac{d^2 \sigma_i^z}{dt^2} \mathbf{v}_i \\ &\quad - \frac{t^3}{3!} \sum_i \frac{d^3 \sigma_i^z}{dt^3} \mathbf{v}_i + \dots \end{aligned} \quad (\text{A11})$$

We find that to zeroth order in the spin-orbit coupling, $\bar{\mathbf{J}}$ coincides with the bare spin current \mathbf{j} in Eq. (3) and the conserved spin current \mathbf{J} in Eq. (2). At leading and higher order, however, the three currents differ.

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